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On the existence of equivalent Dirichlet polynomials whose zeros preserve a topological property

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In this paper, we study the distribution of zeros of the ordinary Dirichlet polynomials which are generated by an equivalence relation introduced by Harald Bohr. Through the use of completely multiplicative functions, we construct equivalent Dirichlet polynomials which have the same critical strip, where all their zeros are situated, and satisfy the same topological property consisting of possessing zeros arbitrarily near every vertical line contained in some substrips inside their critical strip. We also show that the real projections of the zeros of the partial sums of the alternating zeta function, for some particular cases, are dense in their critical intervals.

Keywords: Dirichlet polynomials; Dirichlet character; Multiplicative functions; Bohr's equivalence; Zeros of entire functions.

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1. Introduction

A *general Dirichlet series* is an infinite series that takes the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z},\ a_n \in \mathbb{C}, \tag{1.1}$$

where $\{\lambda_n\}$ is a strictly increasing sequence of non-negative numbers tending to infinity. Associated to it (see [1]), we find the half-planes of convergence $\{z \in \mathbb{C} :$

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$\operatorname{Re} z > \sigma_c\}$ and absolute convergence $\{z \in \mathbb{C} : \operatorname{Re} z > \sigma_a\}$, where σ_c and σ_a are called the abscissas of convergence and absolute convergence respectively.

A classical Dirichlet series is the *Riemann zeta function*, given by

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z},$$

that converges absolutely in $\{z \in \mathbb{C} : \operatorname{Re} z > 1\}$ and admits an analytic continuation over the whole complex plane with only a simple pole at $z = 1$. By using Dirichlet characters, the Riemann zeta function can be generalized to *Dirichlet L-functions*, that are Dirichlet series converging absolutely in $\{z \in \mathbb{C} : \operatorname{Re} z > 1\}$ defined by

$$L_\chi(z) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^z},$$

where $\chi : \mathbb{Z} \to \mathbb{C}$ is any *Dirichlet character* modulo q, with q a positive integer; that is, χ is periodic with period q, $\chi(1) = 1$, $\chi(n) = 0$ when $n \equiv 0 \pmod{q}$ and it is completely multiplicative. Furthermore, by analytic continuation, $L_\chi(z)$ can be extended to a meromorphic function defined on the whole complex plane. All analogues of Riemann hypothesis for the whole class of Dirichlet L-functions are summed up in the so-called *generalized Riemann hypothesis*, which asserts that if the real part of a zero of $L_\chi(z)$ is between 0 and 1, then it is actually $\frac{1}{2}$.

Connected to the Riemann zeta function, another important prototypical Dirichlet series arises from the *alternating zeta function*, which is defined as the analytic continuation of the series $R(z) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$ and converges in $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. The two functions are related for all complex z by the identity $R(z) = (1 - 2^{1-z})\zeta(z)$.

Concerning this subject, a notion of equivalence of two Dirichlet series, $\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ and $\sum_{n=1}^{\infty} b_n e^{-\lambda_n z}$, emerged from Bohr's lifting principle which consisted to associate to a Dirichlet series a power series in infinitely many variables (see [1, Section 8.5] and [3]). Specifically, fixed a Hamel basis $B = \{\beta_k\}$ for the set $\Lambda = \{\lambda_n\}$, which implies that $\lambda_n = \sum_{k=1}^{q_n} r_{n,k}\beta_k$ with the $r_{i,j}$ rational, equivalence of both Dirichet series means that for some sequence $\{y_n\}$ we have

$$b_n = a_n \exp\left(i \sum_{k=1}^{q_n} r_{n,k} y_k\right),\ n \geq 1.$$

Furthermore, by using Kronecker's approximation theorem, Bohr was able to show that equivalent Dirichlet series have the same abscissa of absolute convergence σ_a and take the same set of values in each half-plane $\operatorname{Re} z > \sigma_1$, where $\sigma_1 \geq \sigma_a$ (see Bohr's equivalence theorem in [1, Theorem 8.16]). In this respect, Turan [18, pp. 13-14] used this result to prove that a sufficient condition for the validity of the Riemann hypothesis was the absence of zeros of the partial sums of $\zeta(z)$ in the half-plane $\operatorname{Re} z > 1$. However, this approach was refuted later when it was proved that, for large values of N, the Nth partial sum of $\zeta(z)$ has zeros in the domain $\operatorname{Re} z > 1$ (see for example [11]).

On the other hand, by truncating (1.1) we obtain exponential polynomials called *Dirichlet polynomials.* The zeros of such polynomials constitute the main goal in this paper. It is worth noting that, for purposes of studying such zeros, these exponential polynomials can be normalized to the form

$$P(z) = 1 + \sum_{j=1}^{n} m_j e^{-w_j z},\ n \in \mathbb{N} \tag{1.2}$$

with increasing positive numbers $w_1 < \ldots < w_n$ and $m_j \in \mathbb{C} \setminus \{0\}$, $j = 1, \ldots, n$. Let us recall that the study of the distribution of the zeros of exponential polynomials, or the largest class of almost-periodic functions, is a largely studied topic (see for example [8], [10], [14], [16], [17], [18] or [19]).

Motivated by Bohr's equivalence theorem, which we will adapt to the case of Dirichlet polynomials, this paper studies the existence of some regions, inside the critical strips where all the zeros of some Dirichlet polynomials $P(z)$ are located, verifying the property consisting of having zeros arbitrarily near every vertical line contained in them. That means that the set $\{\operatorname{Re} z : P(z) = 0\}$ is dense in the real interval determined by the infimum and supremum of the real parts of the zeros of $P(z)$ situated on these regions of its critical strip.

In this sense, Farag [7, Theorem 1] showed that for any natural number n sufficiently large the partial sum of the alternating zeta function

$$R_n(z) := \sum_{m=1}^{n} \frac{(-1)^{m-1}}{m^z} \tag{1.3}$$

is a Dirichlet polynomial that has zeros arbitrarily near every vertical line in $S_{(0,1)} = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$.

Also, if $A = \{m_1, m_2, \ldots, m_n\}$ is a set of $n \geq 2$ complex numbers such that $|m_j| = 1$ for $j = 1, 2, \ldots, n$, since $\{\log p_1, \log p_2, \ldots, \log p_n\}$ are linearly independent numbers over the rationals, where $\{p_1, \ldots, p_n\}$ is an ordered set of n prime numbers, then it was proved in [12, Theorem 10] that

$$P_A(z) := 1 + \sum_{j=1}^{n} \frac{m_j}{p_j^z}. \tag{1.4}$$

has zeros arbitrarily near every vertical line in its associated critical strip where all its zeros are located. Previously, for the case that $m_j = 1$ for each $j = 1, 2, \ldots, n$, Moreno [13, p.77] proved that the partial sum $\sum_{j=1}^{n} \frac{1}{p_j^z}$ $(n \geq n_0)$ has zeros arbitrarily near any line on the strip $S_{(0,1)}$.

Along the same line are the works about the closure of the set of the real parts of the zeros of the partial sums of the Riemann zeta function [5]

$$\zeta_n(z) := \sum_{m=1}^{n} \frac{1}{m^z} \tag{1.5}$$

and the zeros of a representative case of nonlattice Dirichlet polynomials given by

$$L_n(z) := 1 - \sum_{m=2}^{n} \frac{1}{m^z}, \tag{1.6}$$

which is related to the theory of fractal strings introduced by Lapidus and Pomerance [6,9].

So our paper considers the following points:

i) In Section 2, we will develop the theory of the Bohr-equivalence for Dirichlet polynomials (see Definition 2.1 and Proposition 2.2) and we will show several representative cases of Bohr-equivalent Dirichlet polynomials.
ii) In Section 3, we will prove that the ordinary Dirichlet polynomials which are Bohr-equivalent have the same density property consisting of having zeros arbitrarily near every vertical line contained in a specific strip (see Theorem 3.1 or Corollary 3.4).
iii) In Section 4, by taking the Dirichlet polynomials $R_n(z)$, $P_A(z)$, $\zeta_n(z)$ and $L_n(z)$ as a basis, our paper also provides a concrete method to generate associated Dirichlet polynomials that preserve the density property on certain strips (see propositions 4.1 and 4.2).
iv) For the partial sums of the alternating zeta function $R_n(z)$, we will improve Farag's results on the density of the real parts of the zeros of $R_n(z)$ for the cases $n = 3, 4, 5$ (see propositions 4.5 and 4.6).

2. Bohr-equivalent Dirichlet polynomials

Consider two Dirichlet polynomials of type (1.2) with the same set of exponents $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, say $f(z) = 1 + \sum_{m=1}^{n} a_m e^{-\lambda_m z}$ and $g(z) = 1 + \sum_{m=1}^{n} b_m e^{-\lambda_m z}$. If $B = \{\beta_1, \beta_2, \ldots, \beta_{q_n}\}$, where q_n depends on n, is a basis for Λ then, for each $m = 1, 2, \ldots, n$, it is verified

$$\lambda_m = \sum_{k=1}^{q_n} r_{m,k}\beta_k, \tag{2.1}$$

for some rational numbers $r_{m,k}$.

Definition 2.1. *We say that two Dirichlet polynomials $f(z) = 1+\sum_{m=1}^{n} a_m e^{-\lambda_m z}$ and $g(z) = 1+\sum_{m=1}^{n} b_m e^{-\lambda_m z}$ of type (1.2) with the same set of exponents are Bohr-equivalent, relative to the basis $\{\beta_1, \beta_2, \ldots, \beta_{q_n}\}$, if for some set of real numbers $Y = \{y_1, y_2, \ldots, y_{q_n}\}$ it is satisfied*

$$b_m = a_m \exp\left(i \sum_{k=1}^{q_n} r_{m,k} y_k\right), \quad \text{for each } m = 1, 2, \ldots, n,$$

where the $r_{m,k}$'s are given by (2.1).

The definition above constitutes a particular case of that existing for the Dirichlet series whose exponents form a strictly increasing sequence of positive numbers (see [1, p.173]). Hence it is a relation of equivalence and it is also independent of the basis (this can be analogously proved as [1, Theorem 8.10 and Theorem 8.11]).

On the other hand, the *ordinary Dirichlet polynomials*, that is those Dirichlet polynomials whose set of exponents Λ is formed by $\{\log 1, \log 2, \dots, \log n\}$, $n \geq 2$, can be written for purposes of zeros as

$$1 + \sum_{m=2}^{n} a_m m^{-z}, \text{ where } a_m \in \mathbb{C} \text{ for each } m = 2, \dots, n \text{ and } a_n \neq 0. \tag{2.2}$$

In this case, the concept of equivalence considered in Definition 2.1 can be reformulated from the following result, which is analogous to that of [1, Theorem 8.12]. However, we will provide its proof for the sake of completeness.

Proposition 2.2. *Two ordinary Dirichlet polynomials of type (2.2),*

$$P(z) = 1 + \sum_{m=2}^{n} a_m m^{-z} \text{ and } Q(z) = 1 + \sum_{m=2}^{n} b_m m^{-z},$$

are Bohr-equivalent if and only if there exists a function $\chi : \{1, 2, \dots, n\} \to \mathbb{C}$ such that

i) $\chi(1) = 1$ *and* $\chi(ml) = \chi(m)\chi(l)$ *for any positive integers* m *and* l *so that* $ml \leq n$*;*
ii) $b_m = \chi(m) a_m$ *for each* $m = 2, \dots, n$*;*
iii) $|\chi(p)| = 1$ *whenever* $a_m \neq 0$ *and* p *a prime divisor of* m*.*

Proof. If $P(z)$ and $Q(z)$ are Bohr-equivalent, by Definition 2.1 there exists a set of real numbers $\{y_1, y_2, \dots, y_{k_n}\}$ such that

$$b_m = a_m \exp\left(i \sum_{j=1}^{k_n} c_{m,j} y_j \right), \text{ for each } m = 2, \dots, n,$$

for some integers $c_{m,j} \geq 0$. Now, define $\chi : \{1, 2, \dots, n\} \to \mathbb{C}$ by the expression

$$\chi(m) = \exp\left(i \sum_{j=1}^{k_n} c_{m,j} y_j \right), \text{ for each } m = 1, 2, \dots, n,$$

then $\chi(1) = 1$, $\chi(ml) = \chi(m)\chi(l)$ for any two integers m, l so that $ml \leq n$, and finally $|\chi(m)| = 1$ and $b_m = \chi(m) a_m$ for each $m = 2, \dots, n$. Hence conditions i), ii) and iii) of the proposition are satisfied.

Conversely, assume that a function $\chi : \{1, 2, \dots, n\} \to \mathbb{C}$ satisfies conditions i), ii) and iii). Take $m \in \{2, \dots, n\}$ so that $a_m \neq 0$, then the complete multiplicativity on the set $\{1, 2, \dots, n\}$ implies that $\chi(m) = \prod_{j=1}^{k_n} f(m, j)$, where $f(m,j) = \begin{cases} \chi(p_j)^{c_{m,j}} & \text{if } p_j | m \\ 1 & \text{otherwise} \end{cases}$. Moreover, condition iii) implies that $|\chi(p_j)| = 1$ for each prime divisor p_j of m. Consequently, for such prime numbers we have that

$\chi(p_j) = e^{iy_j}$ with $y_j = \text{Arg}(\chi(p_j))$, where Arg denotes the principal argument. For the remaining j (if any) we define $y_j = 0$. Thus $f(m,j) = e^{ic_{m,j}y_j}$ for every $j = 1, 2, \ldots, k_n$ and $\chi(m) = \exp(i \sum_{j=1}^{k_n} c_{m,j} y_j)$. Therefore, Definition 2.1 holds for those m such that $a_m \neq 0$. Finally, note that if $a_m = 0$ then condition ii) also implies that $b_m = 0$ and this case is trivial. □

From Proposition 2.2, we can plainly prove that the successive derivatives of two Bohr-equivalent Dirichlet polynomials are Bohr-equivalent. Also, as a consequence of Proposition 2.2, the Dirichlet polynomials $\zeta_3(z) = 1 + 2^{-z} + 3^{-z}$ and $L_3(z) = 1 - 2^{-z} - 3^{-z}$ are Bohr-equivalent because the function $\chi : \{1, 2, 3\} \to \mathbb{C}$ defined as $\chi(1) = 1$, $\chi(2) = -1$ and $\chi(3) = -1$ verifies the required conditions.

However, we next show that the functions $\zeta_n(z)$ and $L_n(z)$, defined in (1.5) and (1.6) respectively, are not Bohr-equivalent for values of n greater than 3.

Example 2.3. Suppose, by reductio ad absurdum, that $\zeta_n(z) = 1+2^{-z}+\ldots+n^{-z}$ and $L_n(z) = 1 - 2^{-z} - \ldots - n^{-z}$ are Bohr-equivalent for some $n \geq 4$. Thus, by Proposition 2.2, there exists a function $\chi : \{1, 2, \ldots, n\} \to \mathbb{C}$ such that, in particular, $b_k = \chi(k)a_k$ for $k = 2, 3, 4$, where $a_2 = a_3 = a_4 = 1$ and $b_2 = b_3 = b_4 = -1$. Therefore, it is accomplished that $\chi(2) = -1$ and $\chi(4) = -1$. However, from the complete multiplicativity, we have

$$-1 = \chi(4) = \chi(2) \cdot \chi(2) = (-1) \cdot (-1) = 1,$$

which is a contradiction.

In order to generate Bohr-equivalent Dirichlet polynomials, we also will consider Dirichlet characters whose definition we recall here.

Definition 2.4. *Let q be a positive integer. A Dirichlet character modulo q is a function $\chi_q : \mathbb{Z} \to \mathbb{C}$ which is periodic with period q, i.e. $\chi_q(n+q) = \chi_q(n)\ \forall n \in \mathbb{Z}$, completely multiplicative, i.e. $\chi_q(nm) = \chi_q(n)\chi_q(m)\ \forall n, m \in \mathbb{Z}$, and which also satisfies $\chi_q(1) = 1$ and $\chi_q(n) = 0$ whenever $(n, q) > 1$, where (n, q) is the greatest common divisor (gcd) of n and q.*

From Proposition 2.2, it is easy to prove the following statements.

Lemma 2.5. *Fixed a natural number $n \geq 2$, let $P(z) = 1+\sum_{m=2}^{n} a_m m^{-z}$, $a_m \in \mathbb{C}$.*

- *i) If $\chi : \mathbb{N} \to \{-1, 1\}$ is a completely multiplicative function, the ordinary Dirichlet polynomials $P(z)$ and $P_\chi(z) = 1 + \sum_{m=2}^{n} \chi(m)a_m m^{-z}$ are Bohr-equivalent.*
- *ii) If $a_n \neq 0$ and $\chi_p : \mathbb{Z} \to \mathbb{C}$ is a Dirichlet character modulo p with p a prime number greater than n, then the ordinary Dirichlet polynomials $P(z)$ and $P_{\chi_p}(z) = 1 + \sum_{m=2}^{n} \chi_p(m)a_m m^{-z}$ are Bohr-equivalent.*

We next show some completely multiplicative functions with values in $\{-1, 1\}$ which may also be used in Lemma 2.5. It is worth noting that the functions verifying these conditions can be generated by randomly choosing the values of the function at the prime numbers. That is, if the $X(p_j)$, with p_j any prime number, denote independent random variables taking only the values 1 or -1 with equal probability and we define $X(1)=1$ and $X(m)=\prod_{j=1}^{k_m} X(p_j)^{c_{m,j}}$ where $m=\prod_{j=1}^{k_m} p_j^{c_{m,j}}$ (that is, the values at all natural numbers are built out of the values at primes by the completely multiplicative property), then this construction provides us all possible candidate functions of this type. For example, Liouville's function is one of the most known functions that verifies these conditions.

Example 2.6. Liouville's function $\lambda : \mathbb{N} \to \{-1,1\}$ is defined by

$$\begin{cases} \lambda(1) & = & 1 \\ \lambda(p_1^{r_1}\cdot p_2^{r_2}\cdots p_k^{r_k}) & = & (-1)^{r_1+r_2+\ldots+r_k} \end{cases},$$

where the $p_j's$ are prime numbers and the $r_j's$ are integer numbers. It is clearly completely multiplicative and it is not a Dirichlet character. Now, let $\{p_1,p_2,\ldots,p_{k_n}\}$ denote the set of all prime numbers less than or equal to some natural number n. Thus, by using Liouville's function, the Dirichlet polynomials

$$P_{\zeta_n}(z)=1+\sum_{m=1}^{k_n} p_m^{-z} \text{ and } P_{L_n}(z)=1-\sum_{m=1}^{k_n} p_m^{-z}$$

are Bohr-equivalent. More generally, if $P_A(z) = 1+\sum_{m=1}^{k_n} a_m p_m^{-z}$ with $A = \{a_1,a_2,\ldots,a_{k_n}\}$ a set of k_n coefficients in the unit circle, then it is plain that $P_{\zeta_n}(z)$ and $P_A(z)$ are Bohr-equivalent.

In order to show a representative Dirichlet character, let p be a prime number. Thus the *Legendre symbol modulo* p, that we will denote as ψ_p, is defined for each integer number m by

$$\psi_p(m)=\begin{cases} 1 \text{ if } m\equiv x^2 \pmod p \text{ for some } x\not\equiv 0 \pmod p \\ 0 \text{ if } p\mid m \\ -1 \text{ if } m\equiv x^2 \pmod p \text{ is unsolvable} \end{cases}.$$

It is clear that the Legendre symbol modulo p is a completely multiplicative function, it is periodic with period p and it vanishes when $(m,p)>1$. Hence it is a Dirichlet character modulo p. Moreover, it takes the values 1 or -1 when m is a prime number with $m\neq p$.

On the other hand, apart from Liouville's function, there exist more known completely multiplicative functions with values in $\{-1,+1\}$. Let p be a prime number. If m is a natural number, let $\Omega_p(m)$ denote the number of prime factors q of m satisfying $\psi_p(q)=-1$, i.e

$$\Omega_p(m)=\mathrm{Card}\{q : q \text{ is prime, } q|m \text{ and } \psi_p(q)=-1\}.$$

Analogously, let $\Omega'_p(m)$ denote the number of prime factors q of a natural number m with $\psi_p(q)=1$, i.e

$$\Omega'_p(m)=Card\{q : q \text{ is prime, } q|m \text{ and } \psi_p(q)=1\}.$$

Thus the *modified Liouville functions for quadratic non-residues modulo p* are defined as

$$\lambda_p(m) = (-1)^{\Omega_p(m)} \text{ and } \lambda'_p(m) = (-1)^{\Omega'_p(m)}.$$

Fixed a prime number p, by using [4, p.7], the function $\lambda_p : \mathbb{N} \to \{-1, 1\}$ is a completely multiplicative function satisfying $\lambda_p(p) = 1$ and $\lambda_p(q) = \psi_p(q)$ for prime numbers $q \neq p$. Also, the function $\lambda'_p : \mathbb{N} \to \{-1, 1\}$ is a completely multiplicative function such that $\lambda'_p(p) = 1$ and $\lambda'_p(q) = -\psi_p(q)$ for primes $q \neq p$.

3. The density property on substrips contained in the critical strip

Given an ordinary Dirichlet polynomial $P(z) = 1 + \sum_{m=2}^{n} a_m m^{-z}$ of type (2.2), it is known that all zeros of $P(z)$ are located on a vertical strip, called the critical strip of $P(z)$, bounded by the real numbers α_P and β_P, where $\alpha_P := \inf \{\operatorname{Re} z : P(z) = 0\}$ and $\beta_P := \sup \{\operatorname{Re} z : P(z) = 0\}$. These bounds allow us to define an interval $I_P := [\alpha_P, \beta_P]$, called critical interval of $P(z)$. Finally, associated with $P(z)$, we will take the set

$$R_P := \overline{\{\operatorname{Re} z : P(z) = 0\}},$$

on which we are going to focus our attention from now on.

In this sense, a first characterization of the sets $R_P \subset I_P$ is obtained from an *ad hoc* version of [2, Theorem 3.1] to be directly applied to our functions $P(z)$. Specifically, fixed an entire number $n \geq 2$ and an ordinary Dirichlet polynomial $P(z) = 1 + \sum_{m=2}^{n} a_m m^{-z}$, for each $1 \leq m \leq n$ let $\mathbf{c_m} = (c_{m,1}, c_{m,2}, \ldots, c_{m,k_n})$ be the vector with non-negative integer components so that $\log m =< \mathbf{c_m}, \mathbf{p} >$, where $\mathbf{p} = (\log p_1, \log p_2, \ldots, \log p_{k_n})$ and $< \cdot, \cdot >$ denotes the standard inner product in $\mathbb{R}^{k_n}$. If we define the function

$$\begin{aligned} F_P &: \mathbb{R} \times \mathbb{R}^{k_n} \to \mathbb{C} \\ (t, x_1, \ldots, x_{k_n}) &\mapsto 1 + \sum_{m=2}^{n} a_m m^{-t} e^{<\mathbf{c_m}, (x_1, \ldots, x_{k_n})> i}, \end{aligned}$$

then [2, Theorem 3.1] states that $t \in R_P$ if and only if there exists some vector $\mathbf{x} \in \mathbb{R}^{k_n}$ such that $F_P(t, \mathbf{x}) = 0$. The proof of this result is analogous to that of [5, Theorem 1]. We next use this characterization to prove the following important theorem.

Theorem 3.1. *Let $P(z) = 1 + \sum_{m=2}^{n} a_m m^{-z}$ and $Q(z) = 1 + \sum_{m=2}^{n} b_m m^{-z}$ be two Bohr-equivalent ordinary Dirichlet polynomials, then $R_P = R_Q$.*

Proof. Since $P(z)$ and $Q(z)$ are Bohr-equivalent, by Proposition 2.2 there exists a vector of real numbers $\mathbf{y} = (y_1, y_2, \ldots, y_{k_n})$ whose components take part in the definition of the function $\chi : \{1, 2, \ldots, n\} \to \mathbb{C}$ given by

$$\chi(m) = \exp\left(i\sum_{j=1}^{k_n} c_{m,j}y_j\right) = e^{<\mathbf{c_m},\mathbf{y}>i}, \text{ for each } m = 1, 2, \ldots, n, \tag{3.1}$$

with $\mathbf{c_m} = (c_{m,1}, c_{m,2}, \ldots, c_{m,k_n})$ the vector of integer components considered previously. This function verifies $b_m = \chi(m)a_m$ and $|\chi(m)| = 1$ for each $m = 2, \ldots, n$. Now, consider $t \in R_P$. According to [2, Theorem 3.1] (see above), there exists some vector $\mathbf{x} \in \mathbb{R}^{k_n}$ such that $F_P(t, \mathbf{x}) = 1 + \sum_{m=2}^{n} a_m m^{-t} e^{<\mathbf{c_m},\mathbf{x}>i} = 0$. Thus, by taking $\mathbf{z} = \mathbf{x} - \mathbf{y} \in \mathbb{R}^{k_n}$ and using (3.1), we have

$$F_Q(t, \mathbf{z}) = 1 + \sum_{m=2}^{n} b_m m^{-t} e^{<\mathbf{c_m},\mathbf{z}>i} = 1 + \sum_{m=2}^{n} \chi(m)a_m m^{-t} e^{<\mathbf{c_m},\mathbf{x}>i} e^{-<\mathbf{c_m},\mathbf{y}>i} =$$

$$1 + \sum_{m=2}^{n} a_m m^{-t} e^{<\mathbf{c_m},\mathbf{x}>i} = 0$$

and, therefore, $t \in R_Q$. Conversely, if $t \in R_Q$ and $\mathbf{z} \in \mathbb{R}^{k_n}$ is so that $F_Q(t, \mathbf{z}) = 0$, then $\mathbf{x} = \mathbf{z} + \mathbf{y}$ verifies $F_P(t, \mathbf{x}) = 0$ and hence the theorem holds. □

Remark 3.2. The converse of Theorem 3.1 is not true. Indeed, the ordinary Dirichlet polynomials $P(z)$ and $Q(z)$ of [15, Example 13] are not equivalent, but they verify $R_P = R_Q$.

As a direct consequence of Theorem 3.1, it is important to remark that two Bohr-equivalent ordinary Dirichlet polynomials, $P(z)$ and $Q(z)$, have the same critical strip. More so, for each zero $z_0 = \sigma_0 + it_0$ of $P(z)$ and for all $\epsilon > 0$, we can find a zero $z_0' = \sigma_0' + it_0'$ of $Q(z)$ such that $\sigma_0 - \epsilon < \sigma_0' < \sigma_0 + \epsilon$.

We will use a new terminology to define this condition on a concrete substrip contained in the critical strip of an ordinary Dirichlet polynomial.

Definition 3.3. *Let $S_{(a,b)} \equiv \{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ be an open vertical strip contained in the critical strip of a Dirichlet polynomial $P(z)$. We say that $P(z)$ has the density property on the strip $S_{(a,b)}$ (or the $\mathcal{D}_{S_{(a,b)}}$ property) when it has zeros arbitrarily near any vertical line inside the strip $S_{(a,b)}$. That is, given σ_2 with $a < \sigma_2 < b$ and $\varepsilon > 0$, a complex number $z^* = \sigma^* + it^*$ can be found such that $\sigma_2 - \varepsilon < \sigma^* < \sigma_2 + \varepsilon$ and $P(z^*) = 0$.*

Equivalently, note that if $P(z)$ is a Dirichlet polynomial which has the density property on the strip $S_{(a,b)}$, this means that the set $\{\operatorname{Re} z : P(z) = 0\}$ is dense in the interval $[a, b]$, that is, $R_P = [a, b]$. Hence we have proved the following result.

Corollary 3.4. *Let $S_{(a,b)}$ be an open vertical strip where the ordinary Dirichlet polynomial $P(z)$ satisfies the $\mathcal{D}_{S_{(a,b)}}$ property. If $Q(z)$ is an ordinary Dirichlet polynomial such that $P(z)$ and $Q(z)$ are Bohr-equivalent, then $Q(z)$ has the $\mathcal{D}_{S_{(a,b)}}$ property.*

4. Associated Dirichlet polynomials verifying the density property

We will apply Corollary 3.4 in order to obtain ordinary Dirichlet polynomials with the $\mathcal{D}_{S_{(a,b)}}$ property for some strips $S_{(a,b)}$. We first show the density property on all the critical strip of the Dirichlet polynomials $P_A(z)$ considered in (1.4).

Proposition 4.1. *Let $\{p_1, p_2, \ldots, p_{k_n}\}$ denote the set of all prime numbers less than or equal to some natural number n. Thus the Dirichlet polynomials of the form $P_A(z) = 1 + \sum_{m=1}^{k_n} \frac{a_m}{p_m^z}$, where $A = \{a_1, a_2, \ldots, a_{k_n}\}$ is a set of complex numbers in the unit circle, have the density property on all their critical strip $S_{(\alpha_{P_A}, \beta_{P_A})}$.*

Proof. From Example 2.6, we deduce that if A_1 and A_2 are two distinct sets of complex numbers in the unit circle, then $P_{A_1}(z)$ and $P_{A_2}(z)$ are Bohr-equivalent. Furthermore, fixed a set $A = \{a_1, a_2, \ldots, a_{k_n}\}$ of complex numbers so that $|a_m| = 1$ for each $m = 1, 2, \ldots, k_n$, from [12, Theorem 10] applied to the function $P_A(-z) = 1 + \sum_{m=1}^{k_n} a_m e^{z \log p_m}$, we also deduce that $P_A(z)$ has the density property on all its critical strip $S_{(\alpha_{P_A}, \beta_{P_A})}$. $\square$

We next use the truncations $R_n(z)$ of the alternating zeta function, defined in (1.3), and the partial sums of the Riemann zeta function to generate Dirichlet polynomials that preserve the density property on the strip $S_{(0,1)} = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ or on some open strips $S_{(a_j, b_j)}$, respectively.

Proposition 4.2. *Let $n > 2$ be a prime number, $\chi_p : \mathbb{Z} \to \mathbb{C}$ any Dirichlet character modulo p with p a prime number greater than n, and $\chi : \mathbb{N} \to \{-1, 1\}$ a completely multiplicative function.*

i) There exists $n_0 \in \mathbb{N}$ such that for each natural number n greater than or equal to n_0, the ordinary Dirichlet polynomials $R_n(z) = 1 + \sum_{m=2}^{n} \frac{(-1)^{m-1}}{m^z}$, $R_{n,\chi_p}(z) = 1 + \sum_{m=2}^{n} \frac{\chi_p(m)(-1)^{m-1}}{m^z}$ and $R_{n,\chi}(z) = 1 + \sum_{m=2}^{n} \frac{\chi(m)(-1)^{m-1}}{m^z}$ have the density property on the strip $S_{(0,1)}$.

ii) There exist some open strips $S_{(a_j, b_j)}$ where the ordinary Dirichlet polynomials $\zeta_n(z) = 1 + \frac{1}{2^z} + \ldots + \frac{1}{n^z}$, $\zeta_{n,\chi_p}(z) = 1 + \frac{\chi_p(2)}{2^z} + \ldots + \frac{\chi_p(n)}{n^z}$ and $\zeta_{n,\chi}(z) = 1 + \frac{\chi(2)}{2^z} + \ldots + \frac{\chi(n)}{n^z}$ have the $\mathcal{D}_{S_{(a_j, b_j)}}$ property.

Proof. i) Observe that, given a natural number $n \geq 2$ and fixed a prime number $p > n$, the Dirichlet polynomials $R_n(z)$, $R_{n,\chi_p}(z)$ and $R_{n,\chi}(z)$ are Bohr-equivalent by Lemma 2.5. Now, [7, Theorem 1] assures the existence of n_0 such that $R_n(z)$ has the density property on the strip $S_{(0,1)}$ for each $n \geq n_0$. Finally, by Corollary 3.4, fixed $n \geq n_0$ the Dirichlet polynomials $R_{n,\chi_p}(z)$, with $p > n$, and $R_{n,\chi}(z)$ also have the $\mathcal{D}_{S_{(0,1)}}$ property.

ii) Fixed $n > 2$ a prime number, let p be a prime number greater than n. Observe that Lemma 2.5 shows that $\zeta_n(z)$, $\zeta_{n,\chi_p}(z)$ and $\zeta_{n,\chi}(z)$ are Bohr-equivalent. On the other hand, [5, Corollary 3] assures the existence of simple zeros of $\zeta_n(z)$ in a certain vertical strip inside its critical strip. Finally, associated to these simple zeros, from [5, Theorem 4] (see also [16, Corollary 15]) we deduce that there exist some open strips $S_{(a_j,b_j)}$ where $\zeta_n(z)$ has the $\mathcal{D}_{S_{(a_j,b_j)}}$ property and, consequently, by taking Corollary 3.4 into account, $\zeta_{n,\chi}(z)$ and $\zeta_{n,\chi_p}(z)$ also have the $\mathcal{D}_{S_{(a_j,b_j)}}$ property. □

Remark 4.3. We recall that the Legendre symbol, Liouville's function and the modified Liouville functions for quadratic non-residues modulo p (see Section 2) are completely multiplicative functions which can be used in propositions 4.2 and 4.2.

From Farag's work [7, Theorem 1], we can state that there exists a natural number n_0 such that the partial sums of the alternating zeta Riemann function $R_n(z)$ have the density property on the concrete strip $S_{(0,1)} \subset S_{(\alpha_{R_n},\beta_{R_n})}$ for each $n \geq n_0$. We next improve this property for the cases $n = 2, 3, 4, 5$. More so, we prove that $R_n(z)$ has the density property on all its critical strip $S_{(\alpha_{R_n},\beta_{R_n})}$ for the values $n = 2, 3, 4$.

Lemma 4.4. *The ordinary Dirichlet polynomials* $R_n(z) = 1 + \sum_{m=2}^{n} \frac{(-1)^{m-1}}{m^z}$ *and* $L_n(z) = 1 - \sum_{m=2}^{n} \frac{1}{m^z}$ *are Bohr-equivalent for the values* $n = 2, 3, 4, 5$.

Proof. Consider the modified Liouville function $\lambda_2(m) = (-1)^{\Omega_2(m)}$ (see page 8), where $\Omega_2(m)$ is the number of prime factors q of a natural number m such that the Legendre symbol modulo 2 satisfies $\psi_2(q) = 1$. Thus it is clear that $\Omega_2(1) = 0$, $\Omega_2(2) = 0$, $\Omega_2(3) = 1$, $\Omega_2(5) = 1$ and thus $\lambda_2(1) = 1$, $\lambda_2(2) = 1$, $\lambda_2(3) = -1$, $\lambda_2(5) = -1$. Also $\lambda_2(4) = \lambda_2(2\cdot 2) = \lambda_2^2(2) = 1$. Hence the completely multiplicative function $\lambda_2 : \{1, 2, 3, 4, 5\} \to \mathbb{C}$ clearly verifies Proposition 2.2 and the result holds. □

For the case $n = 2$ it is clear that all zeros of $R_2(z) = L_2(z) = 1 - 2^{-z}$ are located on the imaginary axis and its critical strip is reduced to this axis.

Proposition 4.5. *The ordinary Dirichlet polynomials* $R_n(z) = 1 + \sum_{m=2}^{n} \frac{(-1)^{m-1}}{m^z}$ *and* $L_n(z) = 1 - \sum_{m=2}^{n} \frac{1}{m^z}$ *have the density property on all its critical strip* $S_{(\alpha_{R_n},\beta_{R_n})}$ *for* $n = 3$ *and* $n = 4$.

Proof. We first note that, by Lemma 4.4, $R_3(z)$ and $L_3(z)$ are Bohr-equivalent. In the same manner, $R_4(z)$ and $L_4(z)$ are Bohr-equivalent. Furthermore, we deduce from [6, Section 3] that $L_3(z)$ and $L_4(z)$ have the density property on all their critical strip $S_{(\alpha_{L_3},\beta_{L_3})}$ and $S_{(\alpha_{L_4},\beta_{L_4})}$ respectively. Consequently, by taking Corollary 3.4 into account, $R_3(z)$ and $R_4(z)$ satisfy the same property. □

Proposition 4.6. *The ordinary Dirichlet polynomials* $R_5(z) = 1 + \sum_{m=2}^{5} \frac{(-1)^{m-1}}{m^z}$ *and* $L_5(z) = 1 - \sum_{m=2}^{5} \frac{1}{m^z}$ *have the density property on the strips* $S_{(-2.3118,0.563367)}$ *and* $S_{(0.851212,\beta_{R_5})}$.

Proof. The result follows from Lemma 4.4, [6, Theorem 3.2] and Corollary 3.4. □

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